

Functional-analysis based tool for testing quark-hadron duality

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Quark-hadron duality is a key concept in QCD, allowing for the description of physical hadronic observables in terms of quark-gluon degrees of freedom. The modern theoretical framework for its implementation is Wilson's operator product expansion (OPE), supplemented by analytic extrapolation from large Euclidean momenta, where the OPE is defined, to the Minkowski axis, where observable quantities are defined. Recently, the importance of additional terms in the expansion of QCD correlators near the Minkowski axis, responsible for quark-hadron duality violations (DVs), was emphasized. In this paper we introduce a mathematical tool that might be useful for the study of DVs in QCD. It is based on finding the minimal distance, measured in the L^∞ norm along a contour in the complex momentum plane, between a class of admissible functions containing the physical amplitude and the asymptotic expansion predicted by the OPE. This minimal distance is given by the norm of a Hankel matrix that can be calculated exactly, using as input the experimental spectral function on a finite interval of the timelike axis. We also comment on the relation between the new functional tool and the more commonly used χ^2 -based analysis. The approach is illustrated on a toy model for the QCD polarization function recently proposed in the literature.

I. INTRODUCTION

The application of theoretical predictions calculated in QCD in terms of quarks and gluons to measured hadronic observables is not straightforward and is usually based on assuming the validity of “quark-hadron duality.” This concept, first used in a rather vague sense, became more precise after its modern implementation in the framework of Wilson's operator product expansion (OPE) [1–3]. Quark-hadron duality assumes that the description in terms of the OPE, valid away from the Minkowski axis, can be analytically continued to match with the description in terms of hadrons, which live on the Minkowski axis.¹

In reality, quark-hadron duality is violated. This was discussed first in detail in Refs. [1–3, 5–8] and more recently in Refs. [9–13]. The violation of duality implies the presence of new terms, in addition to the standard OPE of QCD Green functions, which are necessary in order to obtain agreement with experiment. In principle, the necessity of such terms is not a surprise: since one tries to describe a function known to satisfy a dispersion relation with hadronic unitarity thresholds within a framework that uses quarks and gluons as degrees of freedom, the description cannot be exact, especially near the timelike axis in the complex momentum plane. The question is how large these terms are, and whether they can be calculated theoretically. Of course, one expects the duality violating terms to be small in the Euclidean region, far from the unitarity cut, and larger near the timelike axis. Unfortunately, no theoretical calculations

of duality violations (DVs) are available. Therefore, in order to obtain insight into these terms one must resort to phenomenological models. However, the phenomenological extraction of DVs is made difficult by the fact that both the perturbative expansion in QCD as well as the power-suppressed terms in the OPE are expected to be divergent series. So, as long as we deal with truncated series, it is difficult to disentangle genuinely new terms predicted by specific dynamical mechanisms [1, 3, 6] from contributions of the higher-order terms in the standard QCD expansions, which can be quite large since the expansions are divergent.

For our discussion it is crucial to observe that the analyticity properties in the complex momentum plane of the QCD correlation functions calculated with the OPE are different from those of the exact functions. As for example discussed in Ref. [14], in the confined phase of QCD the poles and the branch-points of the true Green functions are generated by physical hadron states, and no singularities related to the underlying quark and gluon degrees of freedom should appear. On the other hand, in perturbation theory the branch-points of the Green's functions are produced by the quarks and gluons appearing in Feynman diagrams. Other, more complicated singularities make their appearance if one goes beyond simple perturbation theory. For example, renormalization group improved expansions have unphysical spacelike singularities (Landau poles), and power corrections involving nonperturbative vacuum condensates exhibit poles at the origin [15]. Moreover, partial resummations of particular terms in the perturbative expansion, like for instance renormalon chains, might introduce other singularities [16]. For a general review of renormalon singularities, see Ref. [17].

The method we develop in this paper distinguishes be-

¹ For an early discussion of quark-hadron duality, see Ref. [4].

tween the exact Green's function (which has the correct analyticity properties), for which we have incomplete information available from experiment, and the approximant provided by theory, which has different analytical properties. As we shall show below, the difference between these two functions, measured in a suitable norm along a contour in the complex momentum plane, must be larger than (or equal to) a certain nonnegative number δ_0 , which can be computed by techniques of functional analysis. The value of δ_0 can then be used to distinguish between different theoretical approximants, such as for instance approximants involving only the OPE, and approximants involving the OPE and a model for DVs. A smaller value of δ_0 indicates a better approximant.

The paper is organized as follows: in the next section, starting from the information available about a generic polarization function in QCD, we formulate a suitable functional optimization problem. It leads us to consider the minimum distance, δ_0 , measured in the L^∞ norm along a contour in the complex momentum plane, between the QCD approximant and the class of admissible functions containing the physical amplitude. The solution of the problem is presented in Sec. III. In Sec. IV we discuss the properties of δ_0 in the context of a model for the vector polarization function, proposed in Ref. [7]. In Sec. V we consider in particular the dependence of δ_0 on the strength of the DVs in the same model. In Sec. VI, starting from the solution of the functional problem derived in this paper, we discuss a test based on the L^∞ norm, as a possibly interesting alternative to the usual χ^2 (which is based on the L^2 norm) for discriminating between different DV models. In Sec. VII we present our conclusions.

II. FORMULATION OF THE PROBLEM

We consider a light-quark vector vacuum polarization function $\Pi(s)$ in QCD. From causality and unitarity it is known that the function $\Pi(s)$ is analytic of real type (*i.e.*, $\Pi(s^*) = [\Pi(s)]^*$) in the complex s -plane cut along the positive real axis above the lowest threshold ($4m_\pi^2$) for hadron production. These properties are implemented in the once subtracted dispersion relation of the form

$$\Pi(s) = \Pi(0) + \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\sigma(s')}{s'(s' - s - i\epsilon)} ds', \quad (1)$$

where the limit $\epsilon \rightarrow 0^+$ is always understood. We will assume that the spectral function

$$\sigma(s) \equiv \text{Im } \Pi(s + i\epsilon) \quad (2)$$

is known experimentally on a limited energy range, $4m_\pi^2 \leq s \leq s_0$. For instance, the spectral functions of both vector and axial currents of light quarks are measured in the hadronic τ decays up to $s_0 = m_\tau^2$ [18–20]. For the use of hadronic τ decays for the determination of the strong coupling α_s , see Refs. [9, 10, 18–32].

At large momenta sufficiently away from the timelike cut, $\Pi(s)$ can be calculated in the framework of the OPE, in terms of nonperturbative condensates and perturbative coefficients [15]. As discussed recently [1–3, 6–8], additional terms are required beyond the OPE in order to improve the description near the timelike axis. Including these duality violating terms, the QCD prediction is written as

$$\Pi_{\text{QCD}}(s) = \Pi_{\text{OPE}}(s) + \Pi_{\text{DV}}(s), \quad (3)$$

where the OPE contribution can be separated into a pure perturbative (dimension $D = 0$) part and the power-corrections (PC):

$$\Pi_{\text{OPE}}(s) = \Pi_{\text{pert}}(s) + \Pi_{\text{PC}}(s). \quad (4)$$

As discussed above, the exact function $\Pi(s)$ cannot coincide with $\Pi_{\text{QCD}}(s)$ even after suitable resummations of the perturbative part. In order to quantify this deviation, we consider the maximum value of the difference between the two functions along a contour Γ in the complex s plane, which intersects the real positive axis at $s = s_0$. In particular, if Γ is the circle $s = s_0 e^{i\theta}$, which is part of the contour shown in Fig. 1, we consider the difference

$$\delta = \sup_{\theta \in (0, 2\pi)} |\Pi(s_0 e^{i\theta}) - \Pi_{\text{QCD}}(s_0 e^{i\theta})|, \quad (5)$$

where $0 \leq \theta \leq 2\pi$ denotes the angle of the ray in the complex plane (the upper and lower edges of the unitarity cut corresponding to $\theta = 0$ and $\theta = 2\pi$, respectively). Even though the exact function $\Pi(s_0 e^{i\theta})$ is not known, we shall prove in the next section that the quantity δ defined above must be greater than or equal to a certain nonnegative quantity which can be explicitly calculated.²

In order to write the problem in a canonical form, we first make a conformal mapping of the contour Γ onto the contour $|z| = 1$ of the plane $z \equiv \tilde{z}(s)$, where $\tilde{z}(0) = 0$ and $\tilde{z}(s_0) = 1$. For the circle shown in Fig. 1, the transformation is trivial,

$$\tilde{z}(s) = \frac{s}{s_0}. \quad (6)$$

Using this change of variable in Eq. (1), the function $\Pi(s)$ can be written as a function of z in the form

$$\Pi(s_0 z) = \frac{z}{\pi} \int_{x_0}^{1+\eta} dx \frac{\sigma(s_0 x)}{x(x - z - i\epsilon)} + g(z), \quad (7)$$

with $x_0 = 4m_\pi^2/s_0$, and where $g(z)$ is an unknown function, real analytic in the unit disk $|z| \leq 1$. Then, if we define the function $h(\zeta)$ for $\zeta = \exp(i\theta)$ as

$$h(\zeta) = -\frac{\zeta}{\pi} \int_{x_0}^{1+\eta} dx \frac{\sigma(s_0 x)}{x(x - \zeta - i\epsilon)} + \Pi_{\text{QCD}}(s_0 \zeta), \quad (8)$$

² Similar functional-analysis methods were applied for scattering amplitudes in particle physics in Refs. [33], while in the context of QCD such techniques have been applied in Refs. [34].

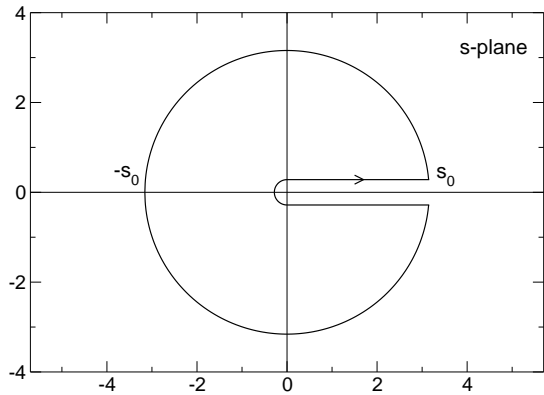


FIG. 1: Analytic properties of $\Pi(s)$ in the disk $|s| < s_0$ of the complex $s = q^2$ plane. $\Pi(s)$ is analytic for all s except $4m_\pi^2 \leq s < \infty$ on the positive real axis.

h is bounded on the circle $|\zeta| = 1$,³ and Eq. (5) can be written as

$$\delta = \|g - h\|_{L^\infty}, \quad (9)$$

where we introduced the L^∞ norm of a function, defined as the supremum of the modulus along the boundary $|z| = 1$:

$$\|F\|_{L^\infty} \equiv \sup_{\theta \in (0, 2\pi)} |F(e^{i\theta})|. \quad (10)$$

The norm δ does not depend on the value of the positive number η , because a variation in its value gets absorbed into a change of the function g .

Starting from Eq. (9), we now consider the following functional extremal problem: find

$$\delta_0 = \min_{g \in H^\infty} \|g - h\|_{L^\infty} \leq \delta, \quad (11)$$

where the minimization is with respect to all the functions $g(z)$ analytic in the disk $|z| < 1$ and bounded on its boundary (this class of functions is denoted as H^∞ [35]).

The quantity δ_0 provides the lower bound announced below Eq. (5). In the next section we present the solution of the problem (11) and the algorithm for calculating δ_0 .

III. SOLUTION OF THE EXTREMAL PROBLEM

We shall apply a “duality theorem” in functional optimization theory [35], which replaces the original minimization problem (11) by a maximization problem in a different functional space, denoted as the “dual” space.

The new problem will turn out to be easier than the original one, allowing us to obtain the solution by means of a numerical algorithm.

Let us denote by H^p , $p < \infty$, the class of functions $F(z)$ which are analytic inside the unit disk $|z| < 1$ and satisfy the boundary condition

$$\|F\|_{L^p} \equiv \left[\frac{1}{2\pi} \int_0^{2\pi} |F(e^{i\theta})|^p d\theta \right]^{1/p} < \infty. \quad (12)$$

In particular, H^1 is the Banach space of analytic functions with integrable modulus on the boundary, and H^2 is the Hilbert space of analytic functions with integrable modulus squared. We introduced already the class H^∞ of functions analytic and bounded in $|z| < 1$, for which the L^∞ norm was defined in Eq. (10). The classes H^p and H^q are said to be dual if the relation $1/p + 1/q = 1$ holds [35]. It follows that H^1 and H^∞ are dual to each other, while H^2 is dual to itself.

We now state the duality theorem of interest to us. Let $h(\zeta = \exp(i\theta))$ be an element of L^∞ . Then the following equality holds (see Sec. 8.1 of Ref. [35]):

$$\min_{g \in H^\infty} \|g - h\|_{L^\infty} = \sup_{G \in S^1} \frac{1}{2\pi} \left| \oint_{|\zeta|=1} G(\zeta) h(\zeta) d\zeta \right|, \quad (13)$$

where S^1 denotes the unit sphere of the Banach space H^1 , *i.e.*, the set of functions $G \in H^1$ which satisfy the condition $\|G\|_{L^1} \leq 1$.

We note first that the equality (13) is automatically satisfied if $h(\zeta)$ is the boundary value of an analytic function in the unit disk, since in this case the minimal norm on the left-hand side is zero, and the right-hand side of Eq. (13) vanishes too, by Cauchy’s theorem. The non-trivial case corresponds to a function $h(\zeta)$ which is not the boundary value of a function analytic in $|z| < 1$. In that case, $h(\zeta)$ will admit the general Fourier expansion

$$h(\zeta) = \sum_{n=0}^{\infty} h_n \zeta^n + \sum_{n=1}^{\infty} c_n \zeta^{-n}, \quad (14)$$

where, due to reality property of $\Pi(s)$ mentioned above, the coefficients h_n and c_n are real.⁴ The analytic continuation of the expansion (14) inside $|z| < 1$ will contain both an analytic part (the first sum) and a nonanalytic part (the second sum). Intuitively, we expect the minimum norm in Eq. (13) to depend explicitly only on the nonanalytic part, *i.e.*, on the coefficients c_n . The proof given below will confirm this expectation.

In order to evaluate the supremum on the right-hand side of Eq. (13) we use a factorization theorem (see the proof of Theorem 3.15 in Ref. [35]) according to which

³ This is the reason we introduced the positive number η in Eq. (7).

⁴ This property will be valid for the coefficients of all expansions below.

every function $G(z)$ belonging to the unit sphere S^1 of H^1 can be written as

$$G(z) = w(z)f(z), \quad (15)$$

where the functions $w(z)$ and $f(z)$ belong to the unit sphere S^2 of H^2 , *i.e.*, are analytic and satisfy the conditions

$$\|w\|_{L^2} \leq 1, \quad \|f\|_{L^2} \leq 1. \quad (16)$$

Therefore, if one writes the Taylor expansions

$$w(z) = \sum_{n=0}^{\infty} w_n z^n, \quad f(z) = \sum_{m=0}^{\infty} f_m z^m, \quad (17)$$

the coefficients satisfy the conditions

$$\sum_{n=0}^{\infty} w_n^2 \leq 1, \quad \sum_{m=0}^{\infty} f_m^2 \leq 1. \quad (18)$$

After introducing the representation (15) into Eq. (13), we obtain the equivalent relation

$$\min_{g \in H^\infty} \|g - h\|_{L^\infty} = \sup_{w, f \in S^2} \left| \frac{1}{2\pi} \oint_{|\zeta|=1} w(\zeta) f(\zeta) h(\zeta) d\zeta \right|, \quad (19)$$

where the supremum on the right-hand side is taken with respect to the functions w and f with the properties mentioned in Eqs. (17) and (18). By inserting into Eq. (19) the expansions (17) we obtain, after a straightforward calculation

$$\min_{g \in H^\infty} \|g - h\|_{L^\infty} = \sup_{\{w_n, f_m\}} \left| \sum_{m,n=1}^{\infty} \mathcal{H}_{nm} w_{n-1} f_{m-1} \right|. \quad (20)$$

Here the supremum is taken with respect to the sequences w_n and f_m subject to the condition (18), and the numbers

$$\mathcal{H}_{nm} = c_{n+m-1}, \quad n, m \geq 1, \quad (21)$$

define a matrix⁵ \mathcal{H} in terms of the negative frequency Fourier coefficients c_n of the function h expanded in Eq. (14), which are calculated as

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} h(e^{i\theta}) d\theta = \frac{1}{2\pi i} \oint_{|\zeta|=1} \zeta^{n-1} h(\zeta) d\zeta. \quad (22)$$

Thinking of w_{n-1} and $\sum_m \mathcal{H}_{nm} f_{m-1}$ as the components of vectors \mathbf{w} and $\mathcal{H}\mathbf{f}$, the absolute value of the sum in

Eq. (20) can be written as $|\mathbf{w} \cdot \mathcal{H}\mathbf{f}|$, and the Cauchy–Schwarz inequality implies that it satisfies

$$|\mathbf{w} \cdot \mathcal{H}\mathbf{f}| \leq \|\mathbf{w}\|_{L^2} \|\mathcal{H}\mathbf{f}\|_{L^2} \leq \|\mathcal{H}\mathbf{f}\|_{L^2}. \quad (23)$$

Since Eq. (23) is saturated for $\mathbf{w} \propto \mathcal{H}\mathbf{f}$, it follows that the supremum in Eq. (20) is given by the L^2 norm of \mathcal{H} . The solution of the minimization problem (11) can then be written as

$$\delta_0 = \|\mathcal{H}\|_{L^2} = \|\mathcal{H}\|, \quad (24)$$

where $\|\mathcal{H}\|$ is the spectral norm, given by the square root of the greatest eigenvalue of the positive-semidefinite matrix $\mathcal{H}^\dagger \mathcal{H}$. In numerical calculations [36], the matrix is truncated at a finite order $m = n = N$, and the convergence of the successive approximants with increasing N is checked. By the duality theorem, the initial functional minimization problem (11) was therefore reduced to a rather simple numerical computation. We finally note that, using the expression (8) for the function h , the coefficients (22) can be written as

$$c_n = \frac{1}{\pi} \int_0^1 x^{n-1} \sigma(s_0 x) dx + \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \Pi_{\text{QCD}}(s_0 e^{i\theta}) d\theta, \quad (25)$$

where in the first term we recognize the moments of the spectral function, which in physical applications are known from experimental measurements.⁶

For our further discussion it is of interest to consider also the minimization problem similar to Eq. (11), where the L^∞ norm is replaced by the L^2 norm

$$\delta_2 = \min_{g \in H^2} \|g - h\|_{L^2}. \quad (26)$$

The solution of this problem is obtained easily by noting that the L^2 norm squared of the difference $g - h$ is

$$\|g - h\|_{L^2}^2 = \sum_{n=0}^{\infty} (g_n - h_n)^2 + \sum_{n=1}^{\infty} c_n^2, \quad (27)$$

where g_n are the coefficients of the Taylor expansion at $z = 0$ of the analytic function $g(z)$. The minimum of the right-hand side is reached for $g_n = h_n$, $n \geq 0$, and reduces to the second sum. Therefore, the solution of the problem (26) is

$$\delta_2 = \left[\sum_{n=1}^{\infty} c_n^2 \right]^{1/2}. \quad (28)$$

Using the fact that the L^∞ norm of every function is larger than its L^2 norm, one finds that

$$\delta_0 \geq \delta_2, \quad (29)$$

⁵ Matrices with elements defined in this way are called Hankel matrices [35].

⁶ Note that the first term in Eq. (25) does not depend on η , because the contour integral in Eq. (22) yields zero for $1 < x < 1 + \eta$.

where the inequality is strict, except for the trivial case of a function $h(\zeta)$ whose expansion (14) contains a single term in the nonanalytic part. We shall return to the relation between the quantities δ_0 and δ_2 in Sec. VI.

IV. APPLICATION TO A SPECIFIC MODEL

We shall illustrate the properties of the quantity δ_0 on a model for the vector polarization function, proposed in Ref. [7] (based on an idea in Refs. [2, 3]), where we can calculate everything exactly. We adopted in fact a simplified version of the model, obtained by including the ρ pole into the “Regge tower” of resonances by adjusting the value of m_0 , from Eq. (3.7) of Ref. [7]. Then $\Pi(s)$ is defined by the model function

$$\Pi_{\text{model}}(s) = -\frac{1}{\zeta} \frac{2F^2}{\Lambda^2} \psi\left(\frac{v+m_0^2}{\Lambda^2}\right), \quad (30)$$

in terms of the Euler digamma function $\psi(v) = \Gamma'(v)/\Gamma(v)$, where the variable v is defined as

$$v = \Lambda^2 \left(\frac{-s - i\epsilon}{\Lambda^2} \right)^\zeta, \quad (31)$$

and the parameters have the numerical values

$$\begin{aligned} \zeta &= 0.95, \quad F = 133.8 \text{ MeV}, \\ \Lambda &= 1.189 \text{ GeV}, \quad m_0 = 0.75 \text{ GeV}. \end{aligned} \quad (32)$$

The “OPE expansion” of Π_{model} was obtained in Ref. [7] using the asymptotic expansion of the digamma function, valid for $|v| \gg 1$, $-\pi < \arg(v) < \pi$,

$$\psi(v) = \log v - \frac{1}{2v} - \sum_{n=1}^{\infty} \frac{B_{2n}}{2n v^{2n}}, \quad (33)$$

where B_{2n} are the Bernoulli numbers. Although the expansion (33) converges for no v , truncated sums provide a good approximation for large $|v|$. From Eq. (33) one can derive the expansions of Π_{model} truncated at a finite order N_{OPE} as [7]

$$\Pi_{\text{OPE}}(s) = -\frac{2F^2}{\Lambda^2} C_0 \log\left(\frac{-s}{\Lambda^2}\right) + \sum_{k=1}^{N_{\text{OPE}}} \frac{C_{2k}}{v^k}, \quad (34)$$

where the first term represents the “purely perturbative” part Π_{pert} and the other ones are the higher-dimensional (PC) terms. From Ref. [7]

$$C_0 = 1, \quad C_{2k} = \frac{2}{\zeta} (-1)^k \frac{1}{k} \Lambda^{2k-2} F^2 B_k \left(\frac{m_0^2}{\Lambda^2} \right), \quad (35)$$

where $B_k(x)$ stand for the Bernoulli polynomials.

As in QCD, the OPE expansion (34) is not accurate near the timelike axis, where DVs are expected to have

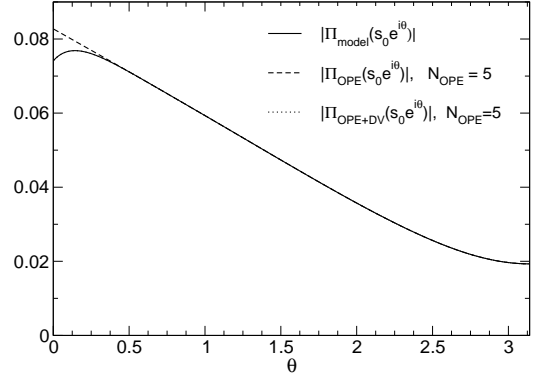


FIG. 2: Modulus of $\Pi_{\text{model}}(s)$ defined in Eq. (30) and of the OPE approximant (34) truncated at $N_{\text{OPE}} = 5$, along the upper semicircle $s = s_0 e^{i\theta}$ with $s_0 = m_\tau^2$. The OPE+DV approximant (37) is not visibly different from the exact function.

a large effect. The description was improved by taking into account the reflection property

$$\psi(v) = \psi(-v) - \pi \cot(\pi v) - \frac{1}{v}, \quad (36)$$

which suggests a modified approximant for $\Pi_{\text{model}}(s)$, valid for large values of $|s|$ and $\text{Re } s > 0$

$$\Pi_{\text{model}}(s) \approx \Pi_{\text{OPE+DV}}(s) \equiv \Pi_{\text{OPE}}(s) + \Pi_{\text{DV}}(s). \quad (37)$$

The correction $\Pi_{\text{DV}}(s)$ is given in the upper half-plane $\text{Im } s > 0$ of the right half-plane $\text{Re } s > 0$ by the expression

$$\Pi_{\text{DV}}(s) = \frac{2\pi F^2}{\Lambda^2 \zeta} \left[-i + \cot \left[\pi \left(\frac{-s}{\Lambda^2} \right)^\zeta + \pi \frac{m_0^2}{\Lambda^2} \right] \right], \quad (38)$$

and can be defined by means of the reality property $\Pi_{\text{DV}}(s^*) = \Pi_{\text{DV}}^*(s)$ in the lower half-plane. In the left half s -plane the correction is assumed to vanish, $\Pi_{\text{DV}}(s) = 0$ for $\text{Re } s \leq 0$.

In order to illustrate the quality of the approximation, we show in Fig. 2 the modulus of the exact function and of its approximants along the upper semicircle $s = s_0 e^{i\theta}$, $\theta \in (0, \pi)$, for $s_0 = m_\tau^2$, for the truncation order $N_{\text{OPE}} = 5$. One can see that the OPE expansion provides a good approximation along the circle except close to the timelike axis, which corresponds to $\theta = 0$. By adding the DV term, the approximant gets very close to the exact function $\Pi_{\text{model}}(s)$, and cannot be distinguished in the figure.

For lower values of s_0 we expect the approximation to deteriorate progressively. For illustration, we show in Fig. 3 the modulus of the exact function and of its approximants along the circle of radius s_0 equal to 1.5 GeV^2 . The deviation between the exact function and the OPE near the timelike axis is larger, but the DV term restores the agreement also in this case.

We will now use the model described above to mimic a typical situation in practical applications to QCD. In the region $s \leq s_0$ we will use as input the spectral function

$$\sigma_{\text{model}}(s) = \text{Im} \Pi_{\text{model}}(s + i\epsilon), \quad s \leq s_0, \quad (39)$$

calculated from the exact model (we show for illustration in Fig. 4 this spectral function in the range from 0 to m_τ^2). Along the circle $|s| = s_0$ we use as input the asymptotic expressions $\Pi_{\text{OPE}}(s)$ or $\Pi_{\text{OPE+DV}}(s)$ defined in Eqs. (34) and (37). Then the Fourier coefficients c_n from Eq. (25) have the specific form

$$c_n = c_n^\sigma + c_n^{\text{OPE}} + c_n^{\text{DV}}, \quad n \geq 1, \quad (40)$$

where

$$c_n^\sigma = \frac{1}{\pi} \int_0^1 x^{n-1} \sigma_{\text{model}}(s_0 x) dx, \quad (41)$$

$$c_n^{\text{OPE}} = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} \Pi_{\text{OPE}}(s_0 e^{i\theta}) d\theta, \quad (42)$$

$$c_n^{\text{DV}} = \frac{1}{2\pi} \int_{\mathcal{R}} e^{in\theta} \Pi_{\text{DV}}(s_0 e^{i\theta}) d\theta. \quad (43)$$

Here \mathcal{R} is the right semicircle, $\theta \in (0, \pi/2] \cup [3\pi/2, 2\pi)$. In practice, due to the reality property of the integrands, to obtain c_n^{OPE} and c_n^{DV} it is enough to integrate only along the upper semicircle and take twice the real part. Having the coefficients c_n , we computed with Mathematica the norm of the Hankel matrix (21), truncated at a finite order N . The convergence was tested by increasing N from 20 to 700. The results presented below are obtained with $N = 500$.

In Table I we give the values of δ_0 for the OPE expansion and the full OPE+DV approximant as a function of the truncation order N_{OPE} appearing in Eq. (34). We took $s_0 = m_\tau^2$. For comparison we also give the difference

$$\delta_{\text{exact}}^{\text{OPE}} = \sup_{\theta \in (0, 2\pi)} |\Pi_{\text{model}}(s_0 e^{i\theta}) - \Pi_{\text{OPE}}(s_0 e^{i\theta})| \quad (44)$$

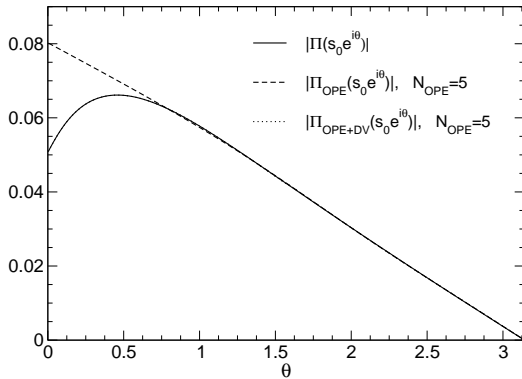


FIG. 3: As in Fig. 2 for $s_0 = 1.5 \text{ GeV}^2$.

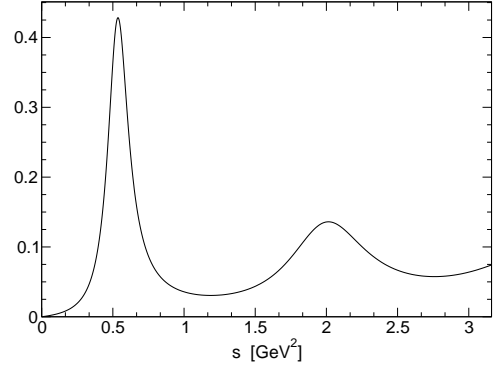


FIG. 4: Spectral function $\sigma_{\text{model}}(s)$ for $s \leq m_\tau^2$.

between the exact function (30) and the OPE expansion (34), and the difference

$$\delta_{\text{exact}}^{\text{OPE+DV}} = \sup_{\theta \in (0, 2\pi)} |\Pi_{\text{model}}(s_0 e^{i\theta}) - \Pi_{\text{OPE+DV}}(s_0 e^{i\theta})| \quad (45)$$

between the exact function and the OPE+DV expression (37). For all cases shown in Table I, the inequality $\delta_0 < \delta_{\text{exact}}$ holds. This is what we expect, in fact: indeed, δ_0 is the minimal value calculated over a class of functions to which the exact function $\Pi_{\text{model}}(s)$ is supposed to belong. One may note that the values of δ_{exact} are not much larger than the lower bound δ_0 in this case.

In the table, we included the value of $N_{\text{OPE}} = 13$ for which $\delta_0^{\text{OPE+DV}}$ and, simultaneously, $\delta_{\text{exact}}^{\text{OPE+DV}}$, take on their minimal values. Actually, it is around this order in the expansion of our model that the OPE starts diverging, for $s_0 = m_\tau^2$. Of course, the fact that the OPE is an asymptotic series has been built into the model. Note that the divergence of the OPE is not visible in δ_0^{OPE} or $\delta_{\text{exact}}^{\text{OPE}}$ (left half of the table) until much larger values of N_{OPE} . These quantities are less sensitive to the OPE because of the missing DV contribution.

TABLE I: Values of δ_{exact} and the lower bound δ_0 computed for the approximants $\Pi_{\text{OPE}}(s)$ and $\Pi_{\text{OPE+DV}}(s)$ on the circle of radius $s_0 = m_\tau^2$, for various truncation orders of the OPE expansion. For $N_{\text{OPE}} > 13$ the expansion starts to diverge.

N_{OPE}	$\delta_{\text{exact}}^{\text{OPE}}$	δ_0^{OPE}	$\delta_{\text{exact}}^{\text{OPE+DV}}$	$\delta_0^{\text{OPE+DV}}$
3	1.98×10^{-2}	6.90×10^{-3}	9.20×10^{-6}	8.73×10^{-6}
5	1.98×10^{-2}	6.89×10^{-3}	1.33×10^{-6}	1.14×10^{-6}
10	1.98×10^{-2}	6.89×10^{-3}	1.97×10^{-7}	1.34×10^{-7}
13	1.98×10^{-2}	6.89×10^{-3}	1.38×10^{-7}	9.31×10^{-8}
20	1.98×10^{-2}	6.89×10^{-3}	3.97×10^{-7}	2.99×10^{-7}
30	1.98×10^{-2}	6.88×10^{-3}	6.90×10^{-5}	6.85×10^{-5}
35	1.90×10^{-2}	6.62×10^{-3}	3.96×10^{-3}	3.95×10^{-3}
40	0.6918	0.6908	0.6918	0.6908

In Table II we give the results obtained for $s_0 =$

TABLE II: As in Table I for $s_0 = 1.5 \text{ GeV}^2$. The divergent character of the expansion becomes manifest at a lower order, for $N_{\text{OPE}} > 7$.

N_{OPE}	$\delta_{\text{exact}}^{\text{OPE}}$	δ_0^{OPE}	$\delta_{\text{exact}}^{\text{OPE+DV}}$	$\delta_0^{\text{OPE+DV}}$
3	4.76×10^{-2}	2.67×10^{-2}	2.20×10^{-4}	1.81×10^{-4}
5	4.75×10^{-2}	2.67×10^{-2}	1.23×10^{-4}	1.01×10^{-4}
7	4.76×10^{-2}	2.67×10^{-2}	1.15×10^{-4}	7.98×10^{-5}
10	4.76×10^{-2}	2.67×10^{-2}	2.45×10^{-4}	1.63×10^{-4}
15	4.76×10^{-2}	2.68×10^{-2}	2.23×10^{-3}	2.09×10^{-3}
20	0.2449	0.2263	0.2272	0.2259

1.5 GeV^2 . The pattern is similar, but the values of both δ_{exact} and the lower bound δ_0 are larger. In this case, the divergent character of the expansion starts manifesting itself beyond $N_{\text{OPE}} = 7$.

V. STRENGTH OF THE DUALITY VIOLATING TERM

In this section we shall argue that the quantity δ_0 may be a useful tool for testing models of DVs in perturbative QCD. We first note that δ_0 quantifies in a certain sense the “non-analyticity” of the input function h defined in terms of the spectral function measured at low energies and some chosen theoretical approximant along the circle in the complex plane, such as the OPE or the OPE plus DVs. If δ_0 is large, the function $h(z)$ defined by this input is far from the class of analytic functions $g(z)$. On the other hand, low values of δ_0 indicate the existence of functions g which are close to h . We can further speculate that, if δ_0 is small, also the difference δ_{exact} between the physical function and the approximant will be small. Tables I and II confirm that δ_0 follows quite closely the values of the exact difference. Therefore, we expect the particular approximant that leads to small values of δ_0 as being favored by the “experimental” input.

In order to check this expectation we investigated the sensitivity of the quantities δ_0 and δ_{exact} to the magnitude of the DV term added to the OPE expansion in the model investigated in the previous section. As a simple exercise, we introduced a strength parameter μ multiplying the DV term, *i.e.*, we replaced Eq. (37) by:

$$\Pi_{\text{OPE+DV}}(s) = \Pi_{\text{OPE}}(s) + \mu \Pi_{\text{DV}}(s). \quad (46)$$

From Eq. (25) it follows that the coefficients (40) are replaced by

$$c_n(\mu) = c_n^\sigma + c_n^{\text{OPE}} + \mu c_n^{\text{DV}}, \quad n \geq 1, \quad (47)$$

and δ_0 calculated as the norm (21) of the Hankel matrix will now be a function of the parameter μ .

In Fig. 5 we show the variations of $\delta_{\text{exact}}^{\text{OPE+DV}}$ and $\delta_0^{\text{OPE+DV}}$ with the parameter μ , taking as before $N_{\text{OPE}} = 5$ and $s_0 = m_\tau^2$. We note that the quantity δ_0 , which

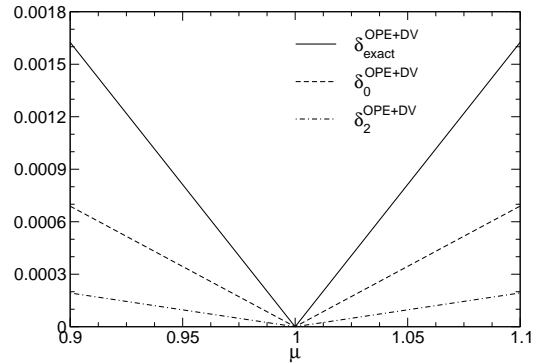


FIG. 5: Dependence on μ of the quantities δ_{exact} and δ_0 calculated for $s_0 = m_\tau^2$. For comparison we present also the quantity δ_2 defined in Eq. (28).

can be computed, in principle, from experimental information available for the QCD correlators, has a behavior similar to that of the exact difference δ_{exact} : both exhibit a sharp minimum at the true value $\mu = 1$. For comparison, we show also the quantity δ_2 defined in Eq. (28). In Fig. 6 we show the three curves for the circle of radius $s_0 = 1.5 \text{ GeV}^2$. We observe that both δ_0 and δ_{exact} exhibit a small plateau near the minimum. As for the quantity δ_2 , it stays below δ_0 in agreement with the exact inequality (29), and is much less sensitive to the variation of the strength parameter μ .

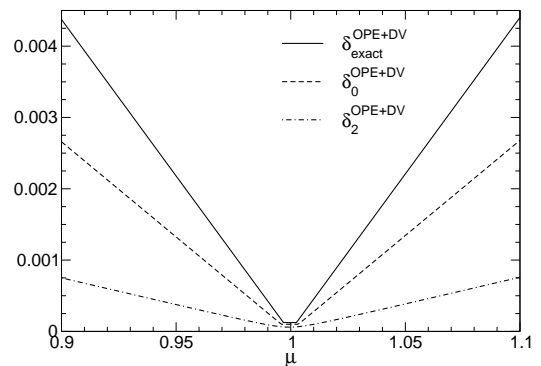


FIG. 6: As in Fig. 5 for $s_0 = 1.5 \text{ GeV}^2$.

VI. DISCUSSION

Within the context of finite-energy sum rules, one exploits analyticity in order to relate QCD predictions to physical measurements by considering a Cauchy integral along the closed contour shown in Fig. 1. Since the exact polarization function is analytic inside the contour,

it satisfies the relation

$$\frac{1}{2\pi i} \oint \phi(s) \Pi(s) ds = 0, \quad (48)$$

where $\phi(s)$ is an arbitrary function holomorphic in the region $|s| \leq s_0$. This relation can be written alternatively as

$$\frac{1}{\pi} \int_0^{s_0} \phi(s) \sigma(s) ds + \frac{1}{2\pi i} \oint_{|s|=s_0} \phi(s) \Pi(s) ds = 0, \quad (49)$$

in terms of the spectral function $\sigma(s)$ defined in Eq. (2). In particular, choosing

$$\phi(s) = \frac{s^{n-1}}{s_0^n}, \quad n = 1, 2, \dots \quad (50)$$

one obtains from Eq. (48) the relations

$$c_n = 0, \quad n = 1, 2, \dots, \quad (51)$$

where c_n are precisely the Fourier coefficients defined in Eq. (25), but where we replace the approximant Π_{QCD} by the exact Π_{model} and use the exact σ_{model} calculated from Eq. (30). This, of course, yields $\delta_0 = 0$.

However, in the example presented in Sec. IV, we used instead of the exact $\Pi_{\text{model}}(s)$ on the circle $|s| = s_0$ its approximants Π_{OPE} or $\Pi_{\text{OPE+DV}}$. Then the coefficients c_n are all different from 0. Through the relations (21) and (24), they produce a nonzero δ_0 , which measures the non-analyticity of the input.

In QCD, the asymptotic expansions of the correlators contain a purely perturbative (dimension-0) part, which has been calculated to order α_s^4 [24], and power corrections containing nonzero vacuum condensates multiplied by logarithmically varying coefficients calculated perturbatively [15]. At the present stage of theoretical knowledge, the power corrections consist of a limited number N_{OPE} of integer powers of $1/s$ with almost constant coefficients. So, unlike in the model considered above, where most of the individual terms in the OPE contribute to a large numbers of moments, in QCD each power correction contributes only to a definite coefficient c_n . Therefore, for the determination of the condensates it is reasonable to use the conditions

$$c_n = 0, \quad n = 1, 2, \dots, N_{\text{OPE}}, \quad (52)$$

defining the so-called “moment finite-energy sum rules.” To further optimize the extraction of the parameters of interest (the strong coupling α_s and the condensates), it is useful to work with the so-called “pinched” moments, defined by using weight functions $\phi(s)$ that vanish near the point s_0 , thus suppressing the contribution of the DV terms [22, 37].

However, when the problem is to discriminate between possible forms for the duality-violating contributions, it is convenient to use weight functions that do not vanish

at $s = s_0$. In recent phenomenological analyses [9, 10], the extraction of the parameters $\{\vec{p}\}$ entering the OPE and a possible *ansatz* for DVs was based on a “fit quality” χ^2 defined as⁷

$$\chi^2(\vec{p}) = \sum_{n \geq 1} \frac{c_n^2(\vec{p})}{\epsilon_n^2}, \quad (53)$$

where

$$\epsilon_n = \delta c_n^\sigma \quad (54)$$

are the errors of the experimental moments. If the errors on the c_n^σ are uncorrelated, this fit quality is the usual χ^2 . The best values of $\{\vec{p}\}$ are found from the maximization of a likelihood function in the parameter space, defined as $\mathcal{L}_2(\vec{p}) \sim \exp[-\chi^2(\vec{p})/2]$. This approach generalizes the strict sum rules (52), allowing for fluctuations within the experimental errors ϵ_n of the coefficients c_n responsible for non-analyticity.

One may view the above likelihood as being induced by the analyticity properties known to be satisfied by the physical function. More formally, we note that χ^2 can be compared to the quantity δ_2 of Eq. (28). If we define a modified version $\bar{\delta}_2(\vec{p})$ of $\delta_2(\vec{p})$ by replacing c_n in Eq. (28) by

$$\bar{c}_n(\vec{p}) = \frac{c_n(\vec{p})}{\epsilon_n}, \quad (55)$$

i.e., the original coefficients $c_n(\vec{p})$ normalized by the experimental errors, we see that

$$\chi^2(\vec{p}) \equiv \bar{\delta}_2^2(\vec{p}). \quad (56)$$

Therefore, in this sense, $\chi^2(\vec{p})$ quantifies the “non-analyticity,” defined here as the minimal distance measured by the L^2 norm on the set H^2 , of a function $h(\zeta)$ having as negative frequency coefficients in the expansion (14), instead of the coefficients c_n , the normalized ratios (55).

Using the comparison between the minimization problems based on the L^∞ norm or the L^2 norm discussed in Sec. III, one might also consider the quantity

$$\bar{\delta}_0^2(\vec{p}) = \|\bar{\mathcal{H}}(\vec{p})\|^2, \quad (57)$$

i.e., the norm squared of a Hankel matrix $\bar{\mathcal{H}}$ constructed from the coefficients $\bar{c}_n(\vec{p})$ by a relation similar to Eq. (21):

$$\bar{\mathcal{H}}_{nm}(\vec{p}) = \bar{c}_{n+m-1}(\vec{p}), \quad n, m \geq 1. \quad (58)$$

We see that $\bar{\delta}_0^2(\vec{p})$ quantifies the non-analyticity of the same function h considered in the χ^2 test, but measured

⁷ Our discussion here is a simplification of the actual analysis performed in Refs. [9, 10].

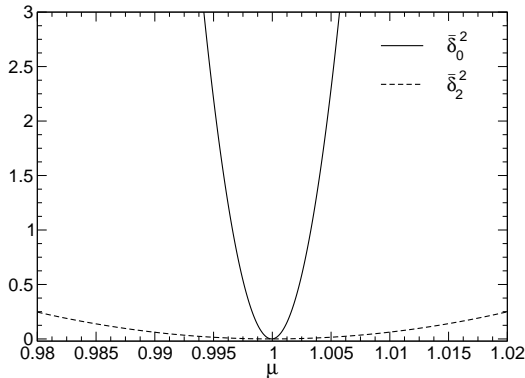


FIG. 7: Variation with μ around the exact value of the quantities $\bar{\delta}_0^2$ and $\bar{\delta}_2^2$, calculated for $s_0 = m_\tau^2$.

now in the L^∞ norm. As above, we may postulate a likelihood $\mathcal{L}_0(\vec{p}) \sim \exp[-\bar{\delta}_0^2(\vec{p})/2]$ in the space of parameters, and determine the best parameters from the minimum of $\bar{\delta}_0^2(\vec{p})$. Since the L^∞ norm tests local properties of the modulus of the function, while the L^2 norm tests the magnitude of the modulus only in the average, we might expect the new test to be stronger and have a greater sensitivity to the parameters that control the non-analyticity of the theoretical input.

In order to illustrate this idea we considered again the model discussed in Sec. IV and generated errors ϵ_n on the coefficients c_n^σ by varying the input spectral function. The experimental data from τ hadronic decays [18–20] have in general larger (relative) errors in the low energy region and near the upper limit $s = m_\tau^2$, and smaller errors in the intermediate region. This implies larger errors δc_n^σ for the higher order coefficients c_n^σ , which are dominated by the large energy region. To simulate this situation, we assumed errors on $\sigma_{\text{model}}(s)$ of 10% for \sqrt{s} below 0.5 GeV and above 1.7 GeV, and of 3% from 0.5 to 1.7 GeV. The errors on c_n^σ were obtained assuming fully correlated errors for $\sigma(s)$ at different energies. The errors ϵ_n thus obtained increase with n , reaching the level of 10% for n around 50. Since this exercise is only for illustrative purposes, we ignore the correlations between c_n^σ and c_m^σ for $n \neq m$.

We then used the quantities $\bar{\delta}_2^2$ and $\bar{\delta}_0^2$ defined in Eqs. (56) and (57), respectively, and checked their sensitivity to the strength parameter μ introduced in Eq. (46). In both cases we have used the same number of coefficients, $N = 500$. As shown in Fig. 7, the variation with μ of $\bar{\delta}_0^2(\mu)$ is much more rapid than that of $\bar{\delta}_2^2(\mu)$. This figure shows how also $\bar{\delta}_0^2(\mu)$ can in principle be used as a potentially interesting fit quality.

As mentioned above, in this exercise we neglected correlations among the coefficients c_n^σ , while in practice one expects these coefficients to be correlated, since they are all computed from the same spectral function. The definitions of $\bar{\delta}_2^2(\vec{p})$ in Eqs. (53–56) and of $\bar{\delta}_0^2(\vec{p})$ in Eq. (57) ig-

nore correlations. Moreover, unlike for the case of $\bar{\delta}_2^2(\vec{p})$, it is not immediately clear how to incorporate them in the L^∞ -norm based test. One could, of course, estimate the error on the parameter μ (or, in general, the fit parameters \vec{p}) by Monte Carlo, taking the full covariance matrix into account. It is of interest to find out which of the two quantities, $\bar{\delta}_2^2(\vec{p})$ or $\bar{\delta}_0^2(\vec{p})$, used in this way, would lead to a smaller error. This is beyond the scope of the present article.

We end this section with the observation that both quantities $\bar{\delta}_2^2$ and $\bar{\delta}_0^2$ depend quadratically on the coefficients c_n . While this is obvious for the L^2 norm, the dependence of $\bar{\delta}_0^2$ on the c_n is not so transparent. We note however that from the definition (10) of the L^∞ norm we expect δ_0 to exhibit an almost piecewise linear dependence on the coefficients c_n and consequently on the parameter μ . This feature is confirmed in Figs. 5 and 6. The quantity $\bar{\delta}_0^2$, defined as the square of δ_0 , is therefore expected to have a piecewise quadratic dependence on the coefficients c_n . Numerical tests show that the dependence of $\bar{\delta}_0^2$ on the higher order coefficients c_n is indeed very close to quadratic. The resulting near-quadratic dependence of $\bar{\delta}_0^2$ on μ is seen in Fig. 7.

VII. CONCLUSIONS AND OUTLOOK

In this article we have argued that the problem of violations of quark-hadron duality in QCD can be investigated with the methods of functional analysis. We showed that the distance, measured in the L^∞ norm along a contour in the complex plane, between an exact QCD correlation function and its theoretical approximation by the OPE plus possible DV terms, must be larger than a certain calculable quantity δ_0 . This quantity is defined by a functional minimization problem, which was solved by a duality theorem in functional optimization. As shown in Sec. III, the problem can be reduced to a numerical algorithm. This allows the calculation of δ_0 as the norm of a Hankel matrix constructed in terms of the Fourier coefficients (22), which can be decomposed as in Eq. (25).

We demonstrated the usefulness of the quantity δ_0 by studying a model proposed in Ref. [7], which allows for a study of the interplay between an asymptotic OPE and duality violations. In this model, we showed that δ_0 is smaller than but quite close to the exact distance, which can also be calculated in this case, but is unknown in a realistic physical situation. This makes δ_0 a potentially suitable tool for the investigation of duality violations, for which an analytic form is not known in general. In particular, we showed that δ_0 is very sensitive to the variation of the parameter μ introduced in Eq. (46) to measure the strength of the duality violating contribution.

We note that with the introduction of the parameter μ , the problem becomes analogous to the search for new physics beyond the Standard Model in experiments at very high energies. There, one tests for the presence of new physics through the “strength parameter” μ of the

signal, while treating the Standard Model as background (for likelihood-based statistical tests used for the discovery of new phenomena see, for instance, Ref. [38]). In our case, we want to detect the presence of DVs (the “new physics”) in addition to the OPE (the “known physics”), which depend on a set of parameters $\vec{\theta}$, often referred to as “nuisance parameters.”

In Sec. VI, we introduced modified versions $\bar{\delta}_0$ and $\bar{\delta}_2$ of δ_0 and a similar distance δ_2 measured in the L^2 norm, to account for the fact that in a realistic application the Fourier coefficients (22) will only be known within certain errors. We showed how $\bar{\delta}_0$ can be computed from a modified Hankel matrix, and we compared the modified quantities to the usual χ^2 commonly used to quantify the distance between data and theory. The standard χ^2 can be interpreted as $\bar{\delta}_2^2$, the minimal distance squared, but measured in the L^2 norm instead of the L^∞ norm. The

discussion in Sec. VI suggests that also $\bar{\delta}_0^2$ can in principle be used as a potentially interesting fit quality, but a practical implementation remains to be explored.

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